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Hopf Bifurcation of a Nonlinear System Derived from Lorenz System Using Centre Manifold Approach

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ABSTRACT

A nonlinear system derived from Lorenz system was presented recently by Tee and Salleh (2013a). Although some basics dynamics were presented, the Hopf bifurcation analysis for the system has not been illustrated yet. Thus, the Hopf bifurcation of the system was determined and the directions of the Hopf bifurcation were demonstrated. We discovered the dynamics on centre manifold of the system and the direction of the Hopf bifurcation in more general cases which supports our finding in Tee and Salleh (2013b). The system shows complexity within the system and is worth to be further researched in the future.

Keywords: Bifurcation, Centre Manifold, Nonlinear Dynamical System.

1. Introduction

In Tee and Salleh (2013a), a nonlinear dynamical system modified from Lorenz system was introduced as follows

$$\begin{cases} \dot{x} = a(by - x); \\ \dot{y} = -xz + cx; \\ \dot{z} = xy - bz; \end{cases} \tag{1}$$

where *x*, *y*, and *z* are variables and *a*, *b*, and *c* are real parameters.

System (1) has a lot of dynamical properties that are yet to be discovered in this particular unique modified system from the original Lorenz system, see Lorenz (1963), Sparrow (1982). The system was newly discovered and caught our interest due to the uniqueness exerted by the system. The sudden changed of the dynamical system flow of trajectories motivate us to further research the system into finding the behaviour of the system through bifurcations. Hence, an extensive research was conducted with the intention of finding the foundation of the individuality by means of bifurcations.

Zhou (2008) did a research based on a new chaotic system modified from Lorenz system which was later named Zhou system. In his paper, he has studied some basic dynamical properties, such as Lyapunov exponents, Poincaré mapping, fractal dimension, bifurcation diagram, continuous spectrum and chaotic dynamical behaviours. Later, Roslan et al. (2010, 2013) studied the Zhou system by discussing the matter for solving Zhou system using Euler's method and fourth-order Runge-Kutta method.

A research was conducted on the dynamical system (1) in Tee and Salleh (2013b) where the Hopf bifurcation for the system (1) was shown by means of first Lyapunov coefficient. The results showed that the system undergoes a subcritical Hopf bifurcation at the equilibrium points P_+ and P_-. However, the research was based on a special case in order to simplify the calculations and it does not represent the whole generality of occurrence of Hopf bifurcation in the system (1).

More research on Hopf bifurcation such as the Hopf bifurcation on Rossler chaotic system with delayed feedback, see Ding et al. (2010), Hopf bifurcation of a unified chaotic system for the generalised Lorenz canonical form (GLCF), see Li et al. (2007) and Hopf bifurcation on a nonlinear three-dimensional dynamical system derived from the Lorenz system, see Zhou et al. (2008), Tigan (2004a, 2004b), Lü et al. (2002).

In this research, we studied in depth on the Hopf bifurcation of the dynamical system by means of centre manifold theorem, see Carr (1981). We presented the dynamics on centre manifold of dynamical system (1). Then, we analysed the direction of the Hopf bifurcation of the system by using the first Lyapunov coefficient, see Kuznetsov (1998).

2. A Modified Lorenz System

From Tee and Salleh (2013a), we have concluded the following results with the basic dynamical properties of the system (1).

Lemma 1

1. If $a > 0$, $b > 0$ and $c < 0$, the dynamical system (1) has only one equilibrium point which is the origin, $P_0(0,0,0)$.

2. If $a > 0$, $b > 0$ and $c > 0$, the dynamical system (1) has three equilibrium points: $P_0(0,0,0)$, $P_+(x_0, y_0, z_0)$ and $P_-(-x_0, -y_0, z_0)$ where $x_0 = b\sqrt{c}$, $y_0 = \sqrt{c}$, $z_0 = c$.

Next, by linearising the dynamical system (1) at the equilibrium point P_+ or P_- yields the following characteristic equation

$$f(\lambda) = \lambda^3 + (a+b)\lambda^2 + (ab + b^2c)\lambda + 2ab^2c = 0. \tag{2}$$

Furthermore, by using the Routh-Hurwitz Criterion

$$\left.\begin{aligned} (a+b) > 0; \\ 2ab^2c > 0; \\ (a+b)(ab+b^2c) - 2ab^2c > 0. \end{aligned}\right\} \tag{3}$$

Lemma 2. The equilibrium points of $P_+(x_0, y_0, z_0)$ and $P_-(-x_0, -y_0, z_0)$ are asymptotically stable if and only if (4) holds.

Then all the coefficients of the equation (3) are all positive. Hence, this leads to $f(\lambda) > 0$ for all $\lambda \geq 0$. Consequently, there is instability in the system (1) if and only if there are two complex conjugate zeros of f. We let $\lambda_1 = i\omega$ and $\lambda_2 = -i\omega$ for some real ω, the sum of three zeros of the function f can be obtained through

$$\lambda_1 + \lambda_2 + \lambda_3 = -(a+b) \tag{4}$$

and we obtained $\lambda_3 = -(a+b)$, which is on the margin of stability when $\lambda_{1,2} = \pm i\omega$. On this margin, we calculated the following equation

$$0 = f\big(-(a+b)\big) = -b[a^2 + ab - abc + b^2c], \tag{5}$$

that is,

$$a_{1,2} = \frac{b(c-1) \pm b\sqrt{c^2 - 6c + 1}}{2}. \tag{6}$$

The stability of the steady state of the equilibrium point $P_+(x_0, y_0, z_0)$ is analyzed by linearising the system (1). Hence, it undergoes linear transformation and becomes as follows

$$\begin{cases} \dot{X} = a(bY - X); \\ \dot{Y} = -(X + b\sqrt{c})Z; \\ \dot{Z} = XY + X\sqrt{c} + b\sqrt{c}Y - bZ. \end{cases} \tag{7}$$

Thus, we obtained the Jacobian matrix at the equilibrium point $P_+(x_0, y_0, z_0)$ as follows

$$J = \begin{bmatrix} -a & ab & 0 \\ 0 & 0 & -b\sqrt{c} \\ \sqrt{c} & b\sqrt{c} & -b \end{bmatrix}. \tag{8}$$

At the equilibrium point by equation (3), Hopf bifurcation will appear when $P_+(x_0, y_0, z_0)$ loses its stability when $(a+b)(ab+b^2c) - 2ab^2c = 0$, that is,

$$cb^2 + a(1-c)b + a^2 = 0, \tag{9}$$

which yields as follow,

$$c = \frac{a(a+b)}{b(a-b)} = c_0. \tag{10}$$

Next, we assuming the equation (2) has only a pair of pure imaginary roots. By substituting $\lambda = \omega i$ into equation (2), the following yields

$$-\omega^3 i - (a+b)\omega^2 + (ab + b^2c)\omega i + 2ab^2c = 0. \tag{11}$$

Then, equation (11) becomes

$$\begin{cases} -\omega^3 + (ab + b^2c)\omega = 0; \\ -(a+b)\omega^2 + 2ab^2c = 0. \end{cases} \tag{12}$$

Later, equation (12) derived a bifurcation surface as follows

$$\begin{cases} a^2 + ab - abc_0 + b^2c_0 = 0; \\ \omega = \sqrt{\frac{2a^2b}{a-b}},\ \frac{2a^2b}{a-b} > 0. \end{cases} \tag{13}$$

Hence, we have obtained our three eigenvalues as follows

$$\begin{cases} \lambda_1 = i\sqrt{\frac{2a^2b}{a-b}}; \\ \lambda_2 = -i\sqrt{\frac{2a^2b}{a-b}}; \\ \lambda_3 = -(a+b). \end{cases} \tag{14}$$

Next, by finding the implicit derivative of λ with respects to c in equation (2), we have

$$\lambda'(c) = \frac{b^2\lambda + 2ab^2}{3\lambda^2 + 2(a+b)\lambda + ab + b^2c}. \tag{15}$$

So, we investigated (15) to determine if Hopf bifurcation occurs by evaluating the following

$$\operatorname{Re}\lambda'(c_0) = \frac{2b^2(a-b)(-2\omega^2a^2 + 3\omega^2ab + 2a^3b - \omega^2b^2)}{(-3\omega^2a + 3\omega^2b + 2a^2b)^2 + (2\omega a^2 - 2\omega b^2)^2} \neq 0, \tag{16}$$

$$\operatorname{Im}\lambda'(c_0) = -\frac{b^2\omega(a-b)(3\omega^2a + 4a^3 - 2a^2b - 4ab^2 - 3\omega^2b)}{(-3\omega^2a + 3\omega^2b + 2a^2b)^2 + (2\omega a^2 - 2\omega b^2)^2} \neq 0, \tag{17}$$

and

$$\operatorname{Im}\lambda(c_0) = \omega = \sqrt{\frac{2a^2b}{a-b}} \neq 0. \tag{18}$$

Theorem 1 [1]. When $c = c_0 > 0$, and $a > 0, b > 0$, $a \neq b$ there is an occurrence of Hopf bifurcation at the equilibrium points $P_+\left(b\sqrt{\frac{a(a+b)}{b(a-b)}}, \sqrt{\frac{a(a+b)}{b(a-b)}}, \frac{a(a+b)}{b(a-b)}\right)$ and $P_-\left(-b\sqrt{\frac{a(a+b)}{b(a-b)}}, -\sqrt{\frac{a(a+b)}{b(a-b)}}, \frac{a(a+b)}{b(a-b)}\right)$ in the system (1).

3. Centre Manifold Theorem

In this section, the dynamics on centre manifold for the dynamical system (1) was found using the method called centre manifold theorem, see Carr (1981). For system (1), the eigenvectors corresponding to the eigenvalues λ_1, λ_2 and λ_3 are respectively

$$V_1 = \begin{pmatrix} 1 \\ \dfrac{a+\omega i}{ab} \\ \dfrac{2ab - a\omega i + b\omega i}{ab(a+b)} \end{pmatrix}, V_2 = \begin{pmatrix} 1 \\ \dfrac{a-\omega i}{ab} \\ \dfrac{2ab + a\omega i - b\omega i}{ab(a+b)} \end{pmatrix},$$

$$V_3 = \begin{pmatrix} -\dfrac{a^2}{a-b} \\ \dfrac{a}{a-b} \\ 1 \end{pmatrix}.$$

Let the eigenvectors be in the form of $m = v_1 + v_2 i$ and $n = v_3$, where v_1, v_2 and v_3 are all real vectors, see Tigan (2004a, 2004b, 2008), Kuznetsov (1998), Craioveanu and Tigan (2008), Wiggins (2000). Hence, using formulae $v_1 = \frac{V_1+V_2}{2}$ and $v_2 = \frac{V_1-V_2}{2i}$ the vectors were obtained as follows

$$v_1 = \begin{pmatrix} 1 \\ \dfrac{1}{b} \\ \dfrac{2}{a+b} \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ \dfrac{\omega}{ab} \\ -\dfrac{\omega(a-b)}{ab(a+b)} \end{pmatrix}, v_3 = \begin{pmatrix} -\dfrac{a^2}{a-b} \\ \dfrac{a}{a-b} \\ 1 \end{pmatrix}. \quad (19)$$

Due to the complexity of the system, an example of substitution is used to present the dynamics of the centre manifold of the system. Let $b = \frac{1}{2}a$. Thus, a matrix, P was form as follows

$$P = \begin{bmatrix} 1 & 0 & -2a \\ \dfrac{2}{a} & \dfrac{2\sqrt{2}}{a} & 2 \\ \dfrac{4}{3a} & -\dfrac{2\sqrt{2}}{3a} & 1 \end{bmatrix}.$$

Then, a linear transformation of the system (1) was performed

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{bmatrix} 1 & 0 & -2a \\ \dfrac{2}{a} & \dfrac{2\sqrt{2}}{a} & 2 \\ \dfrac{4}{3a} & -\dfrac{2\sqrt{2}}{3a} & 1 \end{bmatrix} \begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix},$$

where

$$\begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix} = \begin{bmatrix} \frac{5}{17} & \frac{2}{17}a & \frac{6}{17}a \\ \frac{\sqrt{2}}{34} & \frac{11}{68}\sqrt{2}a & -\frac{9}{34}\sqrt{2}a \\ -\frac{6}{17a} & \frac{1}{17} & \frac{3}{17} \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix},$$

namely

$$\begin{cases} M_1 = \frac{5}{17}X + \frac{2}{17}aY + \frac{6}{17}aZ; \\ M_2 = \frac{\sqrt{2}}{34}X + \frac{11}{68}\sqrt{2}aY - \frac{9}{34}\sqrt{2}aZ; \\ M_3 = -\frac{6}{17a}X + \frac{1}{17}Y + \frac{3}{17}Z. \end{cases}$$

After some tedious calculations, the transformed system was obtained

$$\begin{aligned} \dot{M}_1 &= A(M_1, M_2, M_3) \\ &= \frac{1}{17}(7a\sqrt{2}M_2 + 16a^2M_3 - 6aM_1 + \frac{32}{3}a\sqrt{6}M_1 + \frac{40}{3}a\sqrt{3}M_2 \\ &\quad -7a^2\sqrt{6}M_3 + 12M_1^2 + 12\sqrt{2}M_1M_2 - 12aM_1M_3 - 24a\sqrt{2}M_2M_3 \\ &\quad -24a^2M_3^2), \end{aligned}$$

$$\begin{aligned} \dot{M}_2 &= B(M_1, M_2, M_3) \\ &= -\frac{\sqrt{2}}{408}(24a\sqrt{2}M_2 - 222a^2M_3 - 6aM_1 + 260a\sqrt{6}M_1 + 172a\sqrt{3}M_2 \\ &\quad -75a^2\sqrt{6}M_3 + 216M_1^2 + 216\sqrt{2}M_1M_2 - 216aM_1M_3 \\ &\quad -432a\sqrt{2}M_2M_3 - 432a^2M_3^2), \end{aligned}$$

$$\begin{aligned} \dot{M}_3 &= C(M_1, M_2, M_3) \\ &= \frac{1}{102a}(-18aM_1 - 30a\sqrt{2}M_2 - 105a^2M_3 + 32a\sqrt{6}M_1 + 40a\sqrt{3}M_2 \\ &\quad -21a^2\sqrt{6}M_3 + 36M_{16}^2 + 36\sqrt{2}M_1M_2 - 36aM_1M_3 - 72a\sqrt{2}M_2M_3 \\ &\quad -72a^2M_3^2). \end{aligned} \tag{20}$$

Then, the two-dimensional local centre manifold of the system (1) near the origin is the set

$$W^c_{loc}(0) = \{(M_1, M_2, M_3) \in \mathbb{R}^3 | M_3 = h(M_1, M_2), |M_1| + |M_2| \ll 1\},$$

where

$$h(0,0) = \frac{\partial h}{\partial M_1}(0,0) = \frac{\partial h}{\partial M_2}(0,0) = 0.$$

With the substitution $M_3 = h(M_1, M_2)$ in (3.2), the dynamic on the centre manifold is

$$\begin{aligned}\dot{M}_1 &= A(M_1, M_2, M_3)\\ &= \frac{1}{17}(7a\sqrt{2}M_2 + 16a^2h - 6aM_1 + \frac{32}{3}a\sqrt{6}M_1 + \frac{40}{3}a\sqrt{3}M_2\\ &\quad -7a^2\sqrt{6}h + 12M_1^2 + 12\sqrt{2}M_1M_2 - 12aM_1h - 24a\sqrt{2}M_2h\\ &\quad -24a^2h^2),\end{aligned}$$

$$\begin{aligned}\dot{M}_2 &= B(M_1, M_2, M_3)\\ &= -\frac{\sqrt{2}}{408}(24a\sqrt{2}M_2 - 222a^2h - 6aM_1 + 260a\sqrt{6}M_1 + 172a\sqrt{3}M_2\\ &\quad -75a^2\sqrt{6}h + 216M_1^2 + 216\sqrt{2}M_1M_2 - 216aM_1h - 432a\sqrt{2}M_2h\\ &\quad -432a^2h^2).\end{aligned} \tag{21}$$

Assuming that the function

$$M_3 = h(M_1, M_2) = l_{11}M_1^2 + l_{12}M_1M_2 + l_{22}M_2^2 + \cdots. \tag{22}$$

Substituting $M_1 = p + q, M_2 = i(p - q)$, with $p = \bar{q}$, system (21) becomes the following

$$\dot{p} = P_1 - \frac{i}{408}P_2 \tag{23}$$

where

$$\begin{aligned}P_1 &= \frac{15}{17}q^2 - \frac{3}{17}p^2 - \frac{7}{34}\sqrt{6}a^2h - \frac{4}{17}ap - \frac{2}{17}aq + \frac{107}{204}\sqrt{6}aq + \frac{12}{17}aph\\ &\quad + \frac{7}{68}\sqrt{6}ap - \frac{24}{17}aqh + \frac{8}{17}a^2h - \frac{12}{17}a^2h^2 + \frac{12}{17}pq,\end{aligned}$$

$$\begin{aligned}P_2 &= -252\sqrt{2}p^2 + 36\sqrt{2}q^2 - 180\sqrt{2}aqh + 396\sqrt{2}aph + 111\sqrt{2}a^2h\\ &\quad -216\sqrt{2}pq + 87\sqrt{2}aq - 81\sqrt{2}ap - 420\sqrt{3}ap - 100\sqrt{3}aq\\ &\quad +216\sqrt{2}a^2h^2 + 75\sqrt{3}a^2h,\end{aligned}$$

and

$$\dot{q} = Q_1 + \frac{i}{408} Q_2 \,, \tag{24}$$

where

$$Q_1 = \frac{15}{17}p^2 - \frac{3}{17}q^2 - \frac{7}{34}\sqrt{6}a^2h - \frac{4}{17}aq - \frac{2}{17}ap + \frac{107}{204}\sqrt{6}ap + \frac{12}{17}aqh + \frac{7}{68}\sqrt{6}aq - \frac{24}{17}aph + \frac{8}{17}a^2h - \frac{12}{17}a^2h^2 + \frac{12}{17}pq,$$

$$Q_2 = -252\sqrt{2}q^2 + 36\sqrt{2}p^2 - 180\sqrt{2}aph + 396\sqrt{2}aqh + 111\sqrt{2}a^2h - 216\sqrt{2}pq + 87\sqrt{2}ap - 81\sqrt{2}aq - 420\sqrt{3}aq - 100\sqrt{3}ap + 216\sqrt{2}a^2h^2 + 75\sqrt{3}a^2h.$$

From (22), in the new complex variablesM_3 is in the form of

$$M_3 = L_{11}p^2 + L_{12}pq + L_{22}q^2 + O(|p|^3), \tag{25}$$

with

$$\dot{M}_3 = 2L_{11}\dot{p}p + L_{12}(\dot{p}q + p\dot{q}) + 2L_{22}\dot{q}q + O(|p|^3). \tag{26}$$

From equation (26), we substitute $\dot{p}$ and $\dot{q}$ and obtained the following equation

$$\dot{M}_3 = \sum_{i=1}^{4} R_i - \frac{ia}{408}\sum_{i=1}^{4} S_i + O(|p|^3), \tag{27}$$

where

$$R_1 = -\frac{8}{17}ap^2L_{11} + \frac{7}{34}\sqrt{6}ap^2L_{11} + \frac{107}{102}\sqrt{6}apqL_{11} - \frac{4}{17}apqL_{11} - \frac{7}{17}\sqrt{6}a^2hpL_{11} + \frac{16}{17}a^2hpL_{11}$$

$$R_2 = -\frac{2}{17}ap^2L_{12} + \frac{107}{204}\sqrt{6}ap^2L_{12} - \frac{8}{17}apqL_{12} + \frac{7}{34}\sqrt{6}apqL_{12} - \frac{2}{17}aq^2L_{12} + \frac{107}{102}\sqrt{6}aq^2L_{12}$$

$$R_3 = +\frac{8}{17}a^2hpL_{12} - \frac{7}{34}\sqrt{6}a^2phL_{12} + \frac{8}{17}a^2qhL_{12} - \frac{7}{34}\sqrt{6}a^2qhL_{12} + \frac{16}{17}a^2hqL_{22}$$

$$R_4 = -\frac{8}{17}aq^2hL_{22} + \frac{7}{34}\sqrt{6}aq^2hL_{22} + \frac{107}{102}\sqrt{6}apqL_{22} - \frac{4}{17}apqL_{22} - \frac{7}{17}\sqrt{6}a^2qhL_{22}$$

$$S_1 = -840\sqrt{3}p^2L_{11} + 222\sqrt{2}ahpL_{11} + 150\sqrt{3}ahpL_{11} - 200\sqrt{3}pqL_{11} \\ +174\sqrt{2}pqL_{11}$$
$$S_2 = 75\sqrt{3}ahqL_{12} - 87\sqrt{2}p^2L_{12} - 75\sqrt{3}ahpL_{12} + 100\sqrt{3}p^2L_{12} \\ +87\sqrt{2}q^2L_{12} + 111\sqrt{3}ahqL_{12}$$
$$S_3 = 840\sqrt{3}q^2L_{22} - 222\sqrt{2}ahqL_{22} - 150\sqrt{3}ahqL_{22} + 200\sqrt{3}pqL_{22} \\ -174\sqrt{2}pqL_{22}$$
$$S_4 = -162\sqrt{2}p^2L_{11} - 100\sqrt{3}q^2L_{12} - 111\sqrt{2}ahpL_{12} + 162\sqrt{2}q^2L_{22}.$$

Next, we substituted the equation (25) into equation (20) to obtain the following

$$\dot{M}_3 - \frac{1}{102a}v - \frac{i}{51a}w + O(|p|^3), \tag{28}$$

where

$$v = 18aq + 18ap + 105a^2p^2L_{11} + 21\sqrt{6}a^2pqL_{12} + 105a^2q^2L_{22} \\ +21\sqrt{6}a^2p^2L_{11} + 21\sqrt{6}a^2q^2L_{22} + 105a^2pqL_{12} - 32\sqrt{6}aq \\ -32\sqrt{6}ap - 72pq - 36q^2 - 36p^2,$$
$$w = 15\sqrt{2}ap - 15\sqrt{2}aq + 20\sqrt{3}aq - 20\sqrt{3}ap - 18\sqrt{2}p^2 + 18\sqrt{2}q^2.$$

Then, equating the coefficients of the $p^2, pq,$ andq^2 between the equation (28) and (27), one can find the values of L

$$L_{11} = \frac{1}{a^2}(0.1302693542 - 0.02020385526i),$$

$$L_{12} = \frac{1}{a^2}(0.05600084040 + 1.460777204^{-11}i), \tag{29}$$

$$L_{22} = \frac{1}{a^2}(0.1302693542 + 0.02020385526i),$$

where

$$h = L_{11}p^2 + L_{12}pq + L_{22}q^2 + O(|p|^3).$$

Theorem 2. When $b = \frac{1}{2}a, a > 0, b > 0$, and $c_0 = 6$, the dynamics on the centre manifold of system (1.1) is governed by the equation

$$\dot{p} = P_1 - \frac{i}{408}P_2 \tag{30}$$

at the equilibrium point, $P_+(x_0, y_0, z_0)$ and $P_-(-x_0, -y_0, z_0)$ where

$$P_1 = \frac{15}{17}q^2 - \frac{3}{17}p^2 - \frac{7}{34}\sqrt{6}a^2h - \frac{4}{17}ap - \frac{2}{17}aq + \frac{107}{204}\sqrt{6}aq + \frac{12}{17}aph$$
$$+\frac{7}{68}\sqrt{6}ap - \frac{24}{17}aqh + \frac{8}{17}a^2h - \frac{12}{17}a^2h^2 + \frac{12}{17}pq,$$

$$P_2 = -252\sqrt{2}p^2 + 36\sqrt{2}q^2 - 180\sqrt{2}aqh + 396\sqrt{2}aph + 111\sqrt{2}a^2h$$
$$-216\sqrt{2}pq + 87\sqrt{2}aq - 81\sqrt{2}ap - 420\sqrt{3}ap - 100\sqrt{3}aq$$
$$+216\sqrt{2}a^2h^2 + 75\sqrt{3}a^2h,$$

$$h = L_{11}p^2 + L_{12}pq + L_{22}q^2,$$

while the p and q are conjugate to each other with L_{11}, L_{12} and L_{22} are of (30).

From the Theorem 2, one can obtain the values for g_{20}, g_{11}, g_{02} and g_{21} from the centre manifold of system (1). The values obtained are as follow

$$g_{20} = -0.1950692308 + 0.7825683701i,$$
$$g_{11} = 0.7039940901 - 0.7093248125i,$$
$$g_{02} = 0.8921666156 - 0.2170625130i$$
$$g_{21} = \frac{-0.1317741276 + 0.03293263179i}{a}$$

Next, the first Lyapunov coefficient, see Kuznetsov (1998) is described as follows

$$l_1(0) = \frac{Re\ C_1(0)}{\alpha'(0)},$$
$$C_1(0) = \frac{i}{2\omega}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2},$$

where $\alpha'(0)$ is equation (16). The value obtained is

$$C_1(0) = \frac{-0.2117476583 - 1.033922170i}{a},$$
$$l_1(0) = \frac{14.39884076}{a^2}.$$

Thus, it is clear that the value of $l_1(0) > 0$ for all values of a. One can conclude that the system has a subcritical Hopf bifurcation for all values of a. This result strengthens the previous result that has been done in Tee and Salleh (2013b).

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